

Can Mill's empirical account of arithmetic be defended against the criticisms of Frege?

Can a rational mind derive all of arithmetic from the mere observation of a handful of pebbles? Mill believed this is possible, and Frege thought it absurd. The present essay claims that Mill's account should be treated more sympathetically than hitherto, especially in the light of recent structuralist views of arithmetic, and Lewis's proposal to found our account of mathematics on mereology. We first sketch the relevance of these modern views, then outline Frege's critique, and only at the end expound what Mill actually said, viewed in the light of these approaches.

The fullest account of structuralism is in Shapiro (1997). The main proposal is that arithmetic studies pure relations, rather than of objects. There are platonist structuralists, who believe it is the structures or patterns which are real, rather than the objects, and nominalists (such as Hellman and Chihara), who only accept particulars as real, with structures as just descriptions of what relations are possible or constructible. Also there are those who think of mathematical structures as *sui generis*, and those (such as Quine and Shapiro) who take them to be continuous with the structures of nature. We should expect Mill to be sympathetic to a moderate nominalism, to the continuity of all structures from concrete to abstract, and to the views of the four named philosophers.

Lewis exploited a new development in logic (Boole's plural quantification) to try building mathematics on the science of mereology (Lewis 1993; Varzi 2003). He observed that while mathematics reduces to set theory, it is classes and sub-classes, not sets, which obey the key notion in mereology, that parthood is transitive. Since classes must have members, they avoid the quirk of a necessary set with no members. He therefore builds his account on a reduction of set theory "with the aid of mereology, to the theory of singleton functions" (which resemble traditional 'units' of quantity), so that mathematics is "generalisations" based on classes and their parts. Lewis was a sophisticated modern empiricist, and again we would expect Mill to find this congenial, as he starts from 'members' and the way they fall into 'aggregates'.

Frege's critique of Mill is thorough and perceptive, if not very sympathetic (1884: §7-10, 23-25). Frege was a platonist, viewing numbers as abstract objects. Hence he criticises Mill's empiricism, considering it to be absurd that we could learn huge numbers like 777,864, or basic numbers like 0 and 1, through experience; 0 involves no observations, and 1 seems involved in every observation. Numbers have distinctive characteristics, so knowledge of 777,864 cannot be fully acquired just by extrapolating from a handful of pebbles; if that principle applied, we could learn about two just by studying one. Frege points out that the concept of infinity seems unattainable if it depends on physical objects; indeed, if there were only three objects in existence, then most of arithmetic would vanish. Mill's empirical view also seems rooted in 'psychologistic' subjective observation, rather than in objective truth.

Frege says that Mill has misunderstood the application of arithmetic. While the signs '+' and '=' can indeed be applied to aggregations of pebbles, their meaning is more general than the mere combination and comparison of objects. Mill doesn't understand the difference between parts which make up wholes (pebbles making an aggregate), and parts which are 'logically subordinate' to wholes (big events depending on smaller events).

Mill believes that arithmetic is founded on empirical induction, which Frege rejects. Since numbers have distinctive individuality, Frege compares researching numbers to drilling geological boreholes, where you never know what you might find; induction cannot reveal such information. For example, the position of each number is crucial to its nature, and ordering is intrinsic to numbers. Since Frege takes induction to be essentially probabilistic, this presupposes numbers, rather than founding them. The empiricist view of induction as just a psychological habit of expectation could never give us the objective truth of arithmetic.

He rejects Mill's view that numbers come to us through experience of 'parcels' or 'aggregates' of physical objects, since we can count sounds, or abstractions, or items widely dispersed in space and time. The criticism becomes particularly scathing if Mill believes that aggregates must actually come together in physical space to generate a number. Must the blind people of Germany hold a rally

before they can be counted? How would you count a bundle of straws, if you were allowed to cut them up while rearranging them? We have no conception of how to agglomerate mathematical proofs, or events from different historical eras, yet they can be counted.

Mill regards numbers as properties of physical objects, but Frege says arithmetic is not at all confined to the physical. If numbers are found in physical objects, they must be physical properties, but then it seems to be a category mistake when we count abstractions. We might manage to experience three in contemplating an abstract triangle, but certainly not when contemplating three syllogisms. Frege notes that a pair of boots won't exhibit an experienced numerical property, since it equally exemplifies both one and two.

Finally, Frege takes issue with Mill's concept of a 'unit' (as the concept which enables the observation of number in disparate items). He criticises the concept of a unit at length (1884:§29-44), mainly because he takes the number one to be logically prior to the idea of a unit, so that defining 'one' that way would be circular, and because thinking of something as a unit drains individuality from things, so that they merge, and can no longer be counted. For counting, numbers must be external to objects, not a feature of them.

Since Frege's sustained attack, Mill's views on arithmetic have been held in low esteem. Russell largely ignored him, and logical positivists disregarded Mill because they held arithmetic to be analytic, with only its application an empirical matter. Dummett describes one view of Mill's as "naïve" (a view now described as 'plural quantification') (1991:75), and says that Mill's view of the applicability of deductive reasoning "failed completely" (42). Elsewhere Dummett says that Mill "achieved little more than to point out ..that mathematics can be *applied* to physical reality" (1994). Friend (2007) describes Mill as a "diehard empiricist", and she has a tendency to caricature his views. Hersh says that Frege "trounces" Mill in the *Grundlagen* (1997:142).

Mill was not a *tabula rasa* empiricist; he often cites contributions of the mind to our modes of experience. At times he makes baldly empirical and naturalistic remarks, such as "the fact asserted in the definition of a number is a physical fact" (1843:III.xxiv.5) and "there are no such things as numbers in the abstract. Ten must mean ten bodies..." (II.vi.2), but his view is subtler than this suggests. At no point does he assert that one simply perceives a number when one perceives a group of pebbles. In the act of perception we see that things are of a "kind", that aggregates have "parts", that a part of an aggregate can be viewed as a bare "unit", and that the aggregation can have a "mode of formation". We also compare previous perceptions, and imagine the possibilities of arrangement, decrease and increment in observed aggregates, and Skorupski quotes him as saying that even what is unimaginable to us might still exist (2007:62). Only then does the concept of number come into play, with ensuing deductions, but the empirical story is now quite sophisticated.

Mill's reliance on empirical induction is controversial, but is distinguished from physical science in three ways. First, it is only employed in the earliest stage, to derive a few 'modes of formation' and axioms; after this the subject is entirely deductive. Secondly, he says that this use of induction is quite unconnected with the derivation of causal laws, and that is because (thirdly) the inductions being derived are extremely general (which is why the presence of induction is usually overlooked). Frege treats him as invoking a simple linear sequence of induction (using the successor relation), but what Mill has in mind is (as in physical science) the repeated cross-referencing of the initial inductions. It remains controversial whether the necessity and objectivity of arithmetic can be derived from the apparent contingency of empirical induction, depending on what is meant by 'generality', and our view of nature. Mill's inductive approach certainly has one advantage which platonists must envy – an immediate explanation of the applicability of arithmetic to nature.

Mill's reliance on 'aggregates' draws vociferous attacks. Frege mocks talk of a 'characteristic manner' for arranging an aggregate (and Mill elsewhere talks of its 'proper position'). Undeniably Mill talks in III.xxiv.5 of pebbles being spatially grouped to form aggregates, but I believe Frege has misunderstood. The 'characteristic manner' of arranging objects is some way that makes their total visually obvious. If Frege were obliged to count a bunch of straws, he would first need to identify the 'kind' being counted, and then some arrangement would be needed, either working through them one at a time to count (without duplication), or grouping into small equinumerous groups, with a remainder

at the end. This is clear from Mill's example of the 102/103 horses (III.xxiv.5), where the numbers are facts, but only an arrangement can make this evident to the senses. In this manner Frege's problem with the vastness of 777,864 is dealt with by extrapolations and combinations from graspable quantities. The lessons learned from possible arrangements of small groups is deductively (not physically!) applied to large groups, such as the blind of Germany. Nevertheless, as Skorupski observes (2007:64-5), Mill's account of aggregates will need refinement, perhaps in terms of what is logically possible, or mentally constructible, or conceptually graspable.

Mill is explicit that numbers are properties, and they are always *of* something, but this too is often misunderstood. If there are five pebbles, the handful does not have the property of 'five', as they might be 'heavy'. If numbers were the just properties of pebbles, he says the identity of $2+1$ and 3 would be uninformative, denoting the same property. The key point is that numbers are properties of *magnitudes* (II.vi.2). Objects only have this property if they have a magnitude, and the property is numerical if there are parts (which is where Mill's mereological approach emerges). Thus three refers to *all* the possible 'parcels' made from three items, meaning their possible arrangements. Since magnitudes are generalised features of nature, each property is very widespread, not specific to the pebbles. While the totality of the group is a simple magnitude, the possible arrangements within the parts – an aspect of the numerical property of the magnitude – gives us the mathematical relations (and this is where his structuralism emerges). Algebra is important to Mill because it reveals second-order properties of these magnitudes, in the general relations between the modes of formation of the parts of the magnitude. Unfortunately, in III.xxiv.5 he identifies the property with the 'characteristic manner' of an agglomeration's arrangement. The characteristic manner makes the quantity evident to a mind, but vast agglomerations must have possible arrangements too subtle for any mind to grasp, yet Mill would presumably include these in the numerical property. To make Mill a proto-structuralist we must soften his empiricism when dealing with large structures.

Mill shows his inclination towards structuralism in III.xxiv.5-6, where he adds to the mereological properties of three its place in a larger framework of arithmetical patterns. Putting it simply, 3 is not only $2+1$, but it is also $\sqrt{9}$. Mill outlines this in terms of 'modes of formation', the central idea of his theory. All of arithmetic is said to fall under this heading, as shown in his account of the number 1728. Frege would simply light on this number while 'drilling', but Mill accounts for it as an intersection of many formation-modes, illustrated by the fact that it is both $12 \times 12 \times 12$ and $1000+700+20+8$. The latter is the universal convention for presenting numbers, but Mill's point (absent from Frege's account) is that such a structured presentation is indispensable for large numbers. We must pick one of the modes of formation to present a number (perhaps just the agglomeration of its units). Once again, algebra is crucial for Mill because it leads from the handfuls of pebbles to highly generalised abstract relations between modes of formation (e.g. the binomial theorem). Advanced mathematics is the classification of such functions – with a modal slant, because it investigates *possible* formations, arrangements and structures.

Frege builds his account of numbers around concepts, and rightly argues that this is indispensable to arithmetic, making simplistic empiricism false. But Mill implicitly agrees with this, with his emphasis on 'parts', though he says too little about the wholes which they comprise. Mill has no problem with Frege's pair of boots example, because the identification of what is a part (a unit) precedes any perception of number, and the conceptual identification of a whole precedes the picking out of parts. This approach of Mill's is best labelled as 'mereological', and puts him close to the strategy of Lewis. Thus he observes that

All things possess quantity; consist of parts which can be numbered; and in that character possess all the properties which are called properties of numbers (II.vi.2)

and

Whatever is made up of parts, is made up of the parts of those parts ...And every arithmetical operation is an application of this law... (III.xxiv.5)

He develops his account, using two Euclidean axioms concerning the transitivity of equality, and algebraic second-order generalisation about parts and wholes. The technicalities of this approach only began in 1916 with Lesniewski (Varzi 2003), but Mill was beginning to apply it to arithmetic long

before Lewis. Frege rejects mereological parthood in favour of logical subordination (between events, for example), but that is because he disagrees with Mill over the priority between logic and experience, and he may be influenced by the direction of causation, which Mill considered irrelevant.

Huge problems remain for the Millian view. His discussion is narrow, not even mentioning real or complex numbers, offering no account of zero, showing little interest in simple infinities, and happily ignorant of vast Cantorian ones. He doesn't even give an account of negative numbers, which Friend (2007:132) regards – implausibly – as a huge difficulty. If we look at his initial assumptions about quantity and parts, and allow the addition of conventions, deductions about what is possible, and useful transitory fictions, many of these problems can be plausibly met.

Nevertheless, Mill's original text contains resources to meet most of Frege's objections, and much recent thinking about arithmetic is implicit in Mill's ideas, giving plenty of grounds for a good defence of Mill against Frege's criticisms. It goes too far to present Mill as the 'father of mereological structuralism' (which has yet to be formulated), but he is the father of the naturalistic approach to the foundations of arithmetic, and a figure who deserves respect for his perceptive explorations of how arithmetic might be derived from the physical world.

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